

AN UPPER BOUND FOR THE FIRST HOMOLOGICAL FILLING FUNCTION OF MANIFOLDS OF DIMENSION FOUR

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ABSTRACT. In this paper, we prove that for any closed 4-dimensional Riemannian manifold M with trivial first homology group, if the Ricci curvature $|Ric| \leq 3$, the diameter $\text{diam}(M) \leq D$ and the volume $\text{vol}(M) > v > 0$, then the first homological filling function of M satisfies $HF_1(l) \leq f_1(v, D) \cdot l + f_2(v, D)$, where f_1 and f_2 are some functions that only depend on v and D . As a corollary, the area of a smallest singular minimal surface on M is bounded by $120f_1(v, D) \cdot D + 60f_2(v, D)$.

1. INTRODUCTION

The study of the upper bound for the first homological filling function can be viewed a generalization of the isoperimetric problem in plane. The isoperimetric problem is to determine the largest area of a planar region that can be enclosed by a closed curve with given length. In contrast, the first homological filling function of a Riemannian manifold provides the largest mass of a 2-chain with integer coefficients that fills a 1-cycle with given 1-mass upper bound. In order to describe the main result of our paper, let us first introduce some basic notions of geometric measure theory which is defined in [Fed14].

Definition 1.1. Let M be a Riemannian manifold. The space of singular Lipschitz k -chains in M with coefficient group \mathbb{Z} is $\mathcal{C}_k(M; \mathbb{Z}) = \{\sum_i a_i f_i | f_i : \Delta^k \rightarrow M \text{ a Lipschitz map}, a_i \in \mathbb{Z}\}$. Let $\partial_k : \mathcal{C}_k \rightarrow \mathcal{C}_{k-1}$ be the boundary map. We define the space of k -cycles to be $\mathcal{Z}_k(M; \mathbb{Z}) = \ker \partial_k$.

We define the mass of a singular Lipschitz k -chain to be the following.

Definition 1.2. Let $C = \sum_i a_i f_i \in \mathcal{C}_k(M; \mathbb{Z})$. We define the k -mass of C to be:

$$\text{mass}_k(C) = \sum_i |a_i| \int_{\Delta_k} f_i^*(d \text{vol}_g),$$

where $d \text{vol}_g$ is the Riemannian volume form on M .

Now we define the first homological filling function of a Riemannian manifold M as the following.

Definition 1.3. Let M be a closed Riemannian manifold with trivial first homology group. The first homological filling function $HF_1 : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is defined by

$$HF_1(l) = \sup_{\text{mass}_1(Z) \leq l} \left(\inf_{\partial C = Z} \text{mass}_2(C) \right),$$

where the supremum is taken over all singular Lipschitz 1-cycle $Z \in \mathcal{Z}_1(M; \mathbb{Z})$ with mass bounded by l .

In this paper we prove the following result.

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Theorem 1.4. *Let M be a closed 4-dimensional Riemannian manifold with Ricci curvature $|Ric| \leq 3$, volume $\text{vol}(M) > v > 0$, and diameter $\text{diam}(M) \leq D$. Suppose that the first homology group $H_1(M)$ is trivial, then every Lipschitz singular 1-cycle $C \in \mathcal{Z}_1(M, \mathbb{Z})$ bounds a singular 2-chain in M whose 2-mass is bounded by $f_1(v, D) \cdot \text{mass}_1(C) + f_2(v, D)$, where f_1 and f_2 are functions that only depend on v and D .*

In particular, the first homological filling function HF_1 of M satisfies

$$HF_1(l) \leq f_1(v, D)l + f_2(v, D).$$

Remark 1.5. Note that the functions f_1 and f_2 in the above Theorem 1.4 may not be explicit. In the proof of the above theorem, we used the “bubble tree” construction in the work of J. Cheeger and A. Naber [CN14]. This construction depends on “ ε -regularity” Theorem [CN14, Theorem 2.11]. If one can write down an explicit expression for the constants in the “ ε -regularity” Theorem, then we can write down a formula for the functions f_1 and f_2 . In particular, if M is an Einstein manifold, then by the work [And89] and [And92] of M. Anderson, one can estimate $\varepsilon \sim C \cdot v^{-1/2}$. Note that the work of Cheeger and Naber uses the previous theory of manifolds with Ricci curvature bounded below developed by J. Cheeger and T. Colding [Col97], [CC96], [CC97] and work of M. Anderson [And89], [And92] and Cheeger and Anderson [AC91].

Remark 1.6. The second term $f_2(v, D)$ in the above estimation may not be eliminated. One can consider the following motivating example. We consider a sequence of Riemannian manifolds with boundary $\{(M_i, \partial M_i)\}$ where each M_i is diffeomorphic to the disk bundle over $\mathbb{R}P^2$, equipped with the Eguichi-Hanson metric

$$ds^2 = \left(1 - \frac{a_i}{r^4}\right)^{-1} dr^2 + \frac{r^2}{4} \left(1 - \frac{a_i}{r^4}\right) \sigma_3^2 + \frac{r^2}{4} (\sigma_1^2 + \sigma_2^2),$$

where σ_1, σ_2 and σ_3 are the 1-forms on S^3/\mathbb{Z}^2 , and $a_i \rightarrow 0$. Note that M_i is Ricci flat. The boundary of M_i is a lens space, hence is orientable. One may fill the boundary ∂M_i by a simply-connected 4-manifold and the resulting manifold will have trivial first homology group (cf. [Lic62]). In this case, we consider a generator of the fundamental group of the base space $\mathbb{R}P^2$. The length of this generator $l_i \rightarrow 0$ as $a_i \rightarrow 0$. However, this circle does not bound any 2-chain in the interior of M_i . Therefore, a 2-chain that fills this circle must leave the boundary ∂M_i and the size of this 2-chain is proportional to the diameter of the manifold regardless how small l_i is.

Homological filling function is extensively studied and applied in geometric group theory and Riemannian geometry (cf. [Gro83]). In [NR06], A. Nabutovsky and R. Rotman applied the first homological filling function to bound the smallest area of a singular minimal surface in a Riemannian manifold.

Combined with [NR06, Theorem 1.3], Theorem 1.4 implies following fact about minimal surface in M .

Corollary 1.7. *Let M be a closed 4-dimensional Riemannian manifold with Ricci curvature $|Ric| \leq 3$, volume $\text{vol}(M) > v > 0$, and diameter $\text{diam}(M) \leq D$. Suppose that the first homology group $H_1(M)$ is trivial, then area $A(M)$ of a smallest singular minimal surface in M satisfies*

$$A(M) \leq 60(2f_1(v, D)D + f_2(v, D)).$$

Let us describe the idea of the proof of the above Theorem 1.4. We denote by $\mathcal{M}_1(4, v, D)$, the set of closed 4-dimensional Riemannian manifolds with $H_1(M) = 0$, Ricci curvature $|Ric| \leq 3$, volume $\text{vol}(M) > v > 0$, and diameter $\text{diam}(M) \leq D$.

Suppose that M is a manifold in $\mathcal{M}_1(4, v, D)$ and $C \in \mathcal{Z}_1(M; \mathbb{Z})$ a 1-cycle. We would like to construct a filling of the cycle C whose 2-mass is bounded in terms of $\text{mass}_1(C)$, v , and D . The construction of this filling is based on the results proved by A. Nabutovsky and R. Rotman in [NR03] and J. Cheeger and A. Naber in [CN14].

Let us begin with a simple case where the manifold M is covered by some contractible metric balls of radius r , where r is a fixed constant that is less than the harmonic radius of M . We further assume that the number of the balls in this covering is N . In this case, following from the methods developed in [NR03], one can show that the cycle C bounds a 2-chain in M with 2-mass bounded by $F(N)D \cdot \text{mass}_1(C)$. The idea of the proof in this case is the following. Let \mathfrak{N} be the nerve of the covering of M . We consider a geodesic graph Σ which is obtained by connecting the center of the intersecting balls in the covering by minimizing geodesics. There is a natural map from Σ to the one-skeleton $\mathfrak{N}^{(1)}$ of the nerve. Given a 1-cycle C with $\text{mass}_1(C) = L$. We first look for a 1-cycle C_1 in Σ which is homologous to C such that $C - C_1$ bounds a 2-chain in M with mass $c \cdot Lr$ for some constant c . We then map the 1-cycle C_1 to the nerve \mathfrak{N} to get a cycle C_2 with simplicial length L/r . The estimations in [NR03] show that one can always fill this cycle C_2 in \mathfrak{N} with no more than $F(N) \cdot L/r$ 2-simplices, where F is a function that only depends on the number of the balls N in the covering. Now each simplex in this case essentially corresponds to a filling of a geodesic triangle in Σ whose area is $O(r^2)$. Therefore in this case, one can fill C_1 by a 2-chain in M with mass bounded by $F(N) \cdot L/r \cdot r^2 = F(N)Lr \leq F(N)LD$.

However, in general, it is difficult to cover the manifold M by a certain amount of the contractible metric balls of definite radius. In the case where $M \in \mathcal{M}_1(4, v, D)$, Cheeger and Naber proved in [CN14] that one can cover the manifold by $N(v, D)$ harmonic balls but the radius of these metric balls is not uniformly bounded from below. They further proved that the manifold M decomposes into some body regions \mathcal{B}_j^i and some neck regions \mathcal{N}_j^i , which leads to a finite diffeomorphism type theorem of manifolds in $\mathcal{M}_1(4, v, D)$. The topology of the neck regions is controlled in the sense that $\mathcal{N}_j^i \cong \mathbb{R} \times S^3/\Gamma_j^i$ for some discrete subgroup $\Gamma_j^i \subset O(4)$. (See [CN14, Theorem 8.64]).

Note that in this case if one applies the same argument as above, one may not get any bound for the mass of the 2-chain that fills C_1 . For example, if the manifold is covered by contractible metric balls of two different sizes r and R , where $R \gg r$, then when we map the chain C_1 to \mathfrak{N} , we may get a chain with 1-mass approximately L/r , but the 2-mass of the filling of this chain can be $F(N)L/r \cdot R^2 \sim F(N)LD \cdot (R/r)$, which is not bounded in terms of v, D .

Therefore, in this case, instead of looking for a 1-cycle C_1 in Σ , we are going to decompose the image of C into the bodies \mathcal{B}_j^i and the necks \mathcal{N}_j^i . We will show that one can find a decomposition $C = \sum_{k=1}^N C_k$ such that each 1-cycle C_k either bounds a 2-chain in \mathcal{B}_j^i or it is homologous to a cycle C'_k contained in the consecutive neck region \mathcal{N}_j^i . In the first case, we will use the same argument as above to estimate the 2-mass of a 2-chain that fills C_k , since the regions \mathcal{B}_j^i can be covered by some contractible metric balls of definite size. And in the later case, using the geometry of the region \mathcal{N}_j^i , we show that both $C_k - C'_k$ and C'_k bound some 2-chains in M with controlled 2-mass.

2. PRELIMINARIES

2.1. Finite diffeomorphism type theorem and the bubble tree decomposition.

Let us first introduce the notion of harmonic radius.

Definition 2.1. Let M^n be an n -dimensional Riemannian manifold. We define the harmonic radius $r_h(x)$ at $x \in M$ to be the largest $r > 0$ such that there exists a map $\Phi : B_r(0^n) \rightarrow M$, where $0^n \in \mathbb{R}^n$ is the origin, that satisfies the following:

- (1) Φ is a diffeomorphism onto its image with $\Phi(0^n) = x$.
- (2) $\Delta_g x^l = 0$, $l = 1, \dots, n$, where x^l are the coordinate functions and Δ_g is the Laplace-Beltrami operator.
- (3) If $g_{ij} = \Phi^*(g)$ is the pullback metric on $B_r(0^n)$ and δ_{ij} is the standard Euclidian metric, then

$$\|g_{ij} - \delta_{ij}\|_{C^0, B_r(0^n)} + r \|\partial_k g_{ij}\|_{C^0, B_r(0^n)} \leq 10^{-3}.$$

The above map Φ is called a harmonic coordinate. Note that the third condition above essentially tells us that the map Φ is a Lipschitz map in the following sense.

Lemma 2.2. Let M be a Riemannian manifold and $x \in M$. Suppose $r_h(x) > 0$ is the harmonic radius at x in Definition 2.1 and $\Phi : B_{r_h(x)}(0^n) \rightarrow M$ the harmonic coordinate. For any curve $\gamma : [0, 1] \rightarrow B_{r_h(x)}$ whose image is contained in $B_{r_h(x)}$, the length of the curve $\Phi^{-1}(\gamma)$ satisfies

$$\sqrt{1 - 10^{-3}} \cdot \text{length}(\gamma) \leq \text{length}(\Phi^{-1}(\gamma)) \leq \sqrt{1 + 2 \cdot 10^{-3}} \cdot \text{length}(\gamma).$$

Moreover, we have $B_{r_h/2}(\Phi(0^n)) \subset \Phi(B_{r_h}(0^n))$.

Proof. The proof follows from direct computation. Note that by condition (3) above, the metric $\Phi^*(g)$ on $B_r(0^n)$ satisfies

$$\|g_{ij} - \delta_{ij}\|_{C^0, B_r(0^n)} + r \|\partial_k g_{ij}\|_{C^0, B_r(0^n)} \leq 10^{-3}.$$

If γ is a curve realizing distance from $\partial\Phi(B_{r_h}(0^n))$ to $\Phi(0^n)$, then $\Phi^{-1}(\gamma)$ is a curve connecting $\partial B_{r_h}(0^n)$ with 0^n . Therefore,

$$\text{length}(\gamma) \geq \text{length}(\Phi^{-1}(\gamma)) / \sqrt{1 + 2 \cdot 10^{-3}} \geq r_h / \sqrt{1 + 2 \cdot 10^{-3}} \geq r_h/2.$$

The conclusion follows. \square

In [CN14], Cheeger and Naber proved the finiteness of the number of diffeomorphism type of manifolds M of dimension 4 with $|Ric_M| \leq 3$, $\text{vol}(M) > v > 0$ and $\text{diam}(M) < D$. This theorem is based on the construction of the “bubble tree” decomposition of the manifolds ([CN14, Theorem 8.64]). Their construction is the following.

Up to rescaling, we first cover the manifold M by metric balls $\{B_1(x_i)\}$ such that the balls in $\{B_{1/4}(x_i)\}$ are pairwise disjoint. By a standard volume comparison argument, there are at most $N_0(v, D)$ such balls. In each ball $B_1(x_i)$, there exist scales $r_j^1 > r_0(v, D)$, an integer $N_1 \leq N(v, D)$ and a collection of balls $\{B_{r_j^1}(x_j^1)\}_{j=1}^{N_1}$ such that if $x \in B_1(x_i) \setminus \cup_j B_{r_j^1}(x_j^1)$, then the harmonic radius $r_h(x) \geq r_0(v, D)$, for some function r_0 and N which only depends on v and D . Furthermore, the balls $\{B_{2r_j^1}(x_j^1)\}$ are disjoint. In total, there are at most $N_0 \cdot N$ many such balls. We define the first body $\mathcal{B}^1 = M \setminus \cup_j B_{r_j^1}(x_j^1)$ and the necks $\mathcal{N}_j^2 = B_{2r_j^1}(x_j^1) \cap \mathcal{B}^1$. Note that the manifold $M = \mathcal{B}^1 \cup (\cup_j B_{2r_j^1}(x_j^1))$. As proved in Theorem 8.6 and Lemma 8.40 in [CN14], the geometry of these \mathcal{N}_j^2 are controlled. Namely, there is a diffeomorphism $F : A_{r_j^1/2, 2r_j^1}(0) \rightarrow \mathcal{N}_j^2$, where $A_{r_j^1/2, 2r_j^1}(0)$ is an annulus centered at $0 \in \mathbb{R}^4/\Gamma_j^2$ for some finite discrete subgroup $\Gamma_j^2 \subset O(4)$ such that if $g_{ij} = F^*g$ is the pullback metric,

$$(1) \quad \|g_{ij} - \delta_{ij}\|_{C^0} + r_j^1 \cdot \|\partial_k g_{ij}\|_{C^0} \leq \varepsilon(v).$$

The order of $|\Gamma_j^2|$ is bounded by a function $C(v, D)$.

We repeat this construction to each ball $B_{2r_j^1}(x_j^1)$ and all subsequent metric balls. In general, we have the bodies $\mathcal{B}_i^{k+1} = B_{2r_i^k}(x_j^k) \setminus \cup_j B_{r_j^{k+1}}(x_j^{k+1})$ and the necks $\mathcal{N}_j^{k+1} \subseteq B_{2r_j^k}(x_j^k) \cap \mathcal{B}_i^k$ such that when $x \in \mathcal{B}_i^{k+1}$, then $r_h(x) \geq r_0 \cdot \text{diam}(\mathcal{B}_i^{k+1})$ and $\mathcal{N}_j^{k+1} \cong \mathbb{R} \times S^3/\Gamma_j^{k+1}$, for some $\Gamma_j^{k+1} \subset O(4)$. The reason why this construction ends in finitely many steps is because if for some indices j, k, l , the intersection $\mathcal{N}_l^{k+1} \cap \mathcal{B}_j^k \neq \emptyset$, then $|\Gamma_l^{k+1}| \leq |\Gamma_j^k| - 1$. Therefore, after at most $|\Gamma_j^2| \leq C(v, D)$ many steps, there are no balls in the last bodies. In summary, we have the following decomposition theorem.

Theorem 2.3 ([CN14], Theorem 8.64). *Let M be a 4-dimensional Riemannian manifold with $|\text{Ric}| \leq 3$, $\text{vol}(M) > v > 0$ and $\text{diam}(M) \leq D$. Then M admits a decomposition into bodies and necks*

$$M = \mathcal{B}^1 \cup \bigcup_{j_2=1}^{N_2} \mathcal{N}_{j_2}^2 \cup \bigcup_{j_2=1}^{N_2} \mathcal{B}_{j_2}^2 \cup \dots \cup \bigcup_{j_k=1}^{N_k} \mathcal{N}_{j_k}^k \cup \bigcup_{j_k=1}^{N_k} \mathcal{B}_{j_k}^k,$$

such that the following conditions are satisfied:

- (1) If $x \in \mathcal{B}_i^j$, then $r_h(x) \geq r_0(v, D) \cdot \text{diam}(\mathcal{B}_i^j)$, where r_h is the harmonic radius and r_0 is a constant that only depends on v and D .
- (2) Each \mathcal{N}_i^j is diffeomorphic to $\mathbb{R} \times S^3/\Gamma_i^j$ for some $\Gamma_i^j \subset O(4)$.
- (3) $\mathcal{N}_i^j \cap \mathcal{B}_i^j$ is diffeomorphic to $\mathbb{R} \times S^3/\Gamma_i^j$.
- (4) $\mathcal{N}_i^j \cap \mathcal{B}_{i'}^{j-1}$ is either empty or diffeomorphic to $\mathbb{R} \times S^3/\Gamma_i^j$.
- (5) Each $N_i \leq N(v, D)$ and $k \leq k(v, D)$.

Remark 2.4. The diffeomorphism finiteness result requires some further argument based on the decomposition in Theorem 2.3. One may refer to the proof of Theorem 1.12 in [CN14] for details. The above decomposition theorem is sufficient for us to obtain the estimations in Section 3.

Remark 2.5. From above construction, one can see that the manifold actually decomposes as $M = \mathcal{B}^1 \cup (\cup B_{2r_{j_2}^2}(x_{j_2}^2)) \cup \dots \cup (\cup B_{2r_{j_k}^k}(x_{j_k}^k)) = \mathcal{B}^1 \cup (\cup \mathcal{B}_{j_2}^2) \cup \dots \cup (\cup \mathcal{B}_{j_k}^k)$. The above decomposition is described as the “bubble tree” decomposition in [CN14]. In fact, by Theorem 2.3(4), one can define a tree structure from this decomposition. It is convenient to introduce the following notations for our proof in Section 3.

Definition 2.6. Suppose the manifold M has a decomposition $M = \mathcal{B}^1 \cup (\cup \mathcal{B}_{j_2}^2) \cup \dots \cup (\cup \mathcal{B}_{j_k}^k)$ as in Theorem 2.3 and Remark 2.5. We define a tree structure so that the vertices correspond to the bodies \mathcal{B}_i^j . If for some indices j, i and i' , the intersection $\mathcal{B}_i^j \cap \mathcal{B}_{i'}^{j+1}$ is nonempty, we connect $\mathcal{B}_{i'}^{j+1}$ and \mathcal{B}_i^j with an edge. We denote by L_i^j the set of children of \mathcal{B}_i^j , i.e.,

$$L_i^j = \{\mathcal{B}_k^{j+1} | \mathcal{B}_k^{j+1} \cap \mathcal{B}_i^j \neq \emptyset\}$$

and let

$$\mathcal{L}_i^j = \cup_{\mathcal{B}_k^{j+1} \in L_i^j} \mathcal{B}_k^{j+1}.$$

We denote by \mathcal{T}_i^j the set of vertices of the subtree with root \mathcal{B}_i^j . In other words,

$$\mathcal{T}_i^j = \mathcal{B}_i^j \cup \mathcal{L}_i^j \cup (\cup_{\mathcal{B}_{i_1}^{j_1} \in L_i^j} (\mathcal{L}_{i_1}^{j_1} \cup (\cup_{\mathcal{B}_{i_2}^{j_2} \in L_{i_1}^{j_1}} \mathcal{L}_{i_2}^{j_2} \cup \dots))).$$

Let us end this section by showing some elementary result about the geometry of a neck \mathcal{N}_i^{j+1} in the above decomposition theorem.

Definition 2.7. Consider a diffeomorphism $F : A_{r_i^j/2, 2r_i^j}(0) \rightarrow \mathcal{N}_i^{j+1}$ such that equation (1) is satisfied, where $A_{r_i^j/2, 2r_i^j}(0)$ is an annulus centered at $0 \in \mathbb{R}^4/\Gamma_i^{j+1}$ for some finite discrete subgroup $\Gamma_i^{j+1} \subset O(4)$. The boundary of \mathcal{N}_i^{j+1} consists of 2 connected components S_1 and S_2 , which corresponds to the image of the inner and outer boundary of $A_{r_i^j/2, 2r_i^j}(0)$ under the map F . We define the thickness of the neck \mathcal{N}_i^{j+1} to be

$$\text{thick}(\mathcal{N}_i^{j+1}) = \inf_{x_1 \in S_1, x_2 \in S_2} \text{dist}(x_1, x_2),$$

where $\text{dist}(x_1, x_2)$ is the distance between x_1 and x_2 in the neck \mathcal{N}_i^{j+1} .

Then

Lemma 2.8.

$$\text{diam}(\mathcal{N}_i^{j+1}) \leq B(v) \cdot \text{thick}(\mathcal{N}_i^{j+1}),$$

where $B(v)$ is a function that only depends on v , and $\text{diam}(\mathcal{N}_i^{j+1})$ is the diameter of a neck \mathcal{N}_i^{j+1} as a manifold with pullback metric.

Proof. Let $F : A_{r_i^j/2, 2r_i^j}(0) \rightarrow \mathcal{N}_i^{j+1}$ be a diffeomorphism, where $A_{r_i^j/2, 2r_i^j}(0)$ is an annulus centered at $0 \in \mathbb{R}^4/\Gamma_i^{j+1}$ for some finite discrete subgroup $\Gamma_i^{j+1} \subset O(4)$, such that if $g_{ij} = F^*g$ is the pullback metric, $\|g_{ij} - \delta_{ij}\|_{C^0} + r_i^j \|\partial g_{ij}\|_{C^0} \leq \varepsilon(v)$. Because $\text{diam}(S^3/\Gamma_i^{j+1}) \leq \pi$, we have

$$\text{diam}(A_{r_i^j/2, 2r_i^j}(0)) \leq 2\pi r_i^j + 2(2r_i^j - r_i^j/2) \leq 10r_i^j.$$

Now $\|g_{ij} - \delta_{ij}\|_{C^0} \leq \varepsilon(v)$ implies that

$$\text{thick}(\mathcal{N}_i^{j+1}) \geq \frac{2r_i^j - r_i^j/2}{\sqrt{1 + 2\varepsilon(v)}} = \frac{3r_i^j}{2\sqrt{1 + 2\varepsilon(v)}}$$

and

$$\text{diam}(A_{r_i^j/2, 2r_i^j}(0)) \geq \frac{\text{diam}(\mathcal{N}_i^{j+1})}{\sqrt{1 - \varepsilon(v)}}.$$

Therefore, we obtain that

$$\text{diam}(\mathcal{N}_i^{j+1}) \leq B(v) \cdot \text{thick}(\mathcal{N}_i^{j+1})$$

where $B(v) = \frac{20}{3} \sqrt{1 - \varepsilon(v)} \sqrt{1 + 2\varepsilon(v)}$. □

2.2. Nerve of a covering and geodesic graph. Let $M \in \mathcal{M}_1(4, D, v)$. In this section, we are going to introduce the nerve of a harmonic covering of M as well as a geodesic graph associated to this covering.

Let \mathcal{B}_i^j be a body in the bubble tree decomposition of M as in Theorem 2.3. The harmonic radius of a body is defined to be the infimum over harmonic radii of all points in the body. If $r_{h_i}^j$ is the harmonic radius of body \mathcal{B}_i^j , let

$$R_i^j = r_{h_i}^j/20.$$

By Theorem 2.3 and Remark 2.5, one can cover each body \mathcal{B}_i^j uniformly by no more than $(\frac{\text{constant}}{r_0(v, D)})^4$ balls of radius R_i^j . We denote $\mathcal{O}(\mathcal{B}_i^j)$ the covering of \mathcal{B}_i^j by those (open) balls

of radius R_i^j . We denote $\mathcal{O}(\mathcal{T}_i^j) = \cup_{i,j} \mathcal{O}(B_i^j)$, where the union is over all bodies in \mathcal{T}_i^j . Let $\mathcal{O} = \cup_{i,j} \mathcal{O}(\mathcal{T}_i^j)$. Let \tilde{N} be the number of balls in \mathcal{O} , then

$$\tilde{N} \leq \left(\frac{\text{constant}}{r_0(v, D)} \right)^4 \cdot N(v, D) \cdot k(v, D).$$

We enumerate the centers of those balls as $\{x_1, \dots, x_{\tilde{N}}\}$. For simplicity, if we do not want to emphasize which body x_i belongs to, we denote the harmonic ball centered at x_i with radius R_i^j for some j by $B_{R_i}(x_i)$.

Let us consider the nerve \mathfrak{N} of the covering \mathcal{O} . Let the continuous map $f : M \rightarrow \mathfrak{N}$ be the natural map associated to the nerve. (The reader may refer to [Coo15, Section 4.3] for the definition of the nerve and the natural map). \mathfrak{N} is a simplicial complex with dimension bounded by \tilde{N} . Let $\mathfrak{N}^{(k)}$ be the k -skeleton of \mathfrak{N} . Note that since each \mathcal{T}_i^j is covered by $\mathcal{O}(\mathcal{T}_i^j)$, let $\mathfrak{N}(\mathcal{T}_i^j)$ be the nerve of \mathcal{T}_i^j corresponding to this subcover. Then $\mathfrak{N}(\mathcal{T}_i^j)$ is a simplicial subcomplex of \mathfrak{N} . The natural map from $\mathcal{O}(\mathcal{T}_i^j)$ to $\mathfrak{N}(\mathcal{T}_i^j)$ is defined by the restriction of f to the image of $\mathcal{O}(\mathcal{T}_i^j)$ under the inclusion map.

Next, we define a geodesic graph Γ associated to the covering \mathcal{O} of M . The vertices of Γ are the centers $\{x_1, \dots, x_{\tilde{N}}\}$ of the balls in the covering \mathcal{O} . If the intersection of two balls $B_{R_i}(x_i) \cap B_{R_j}(x_j) \neq \emptyset$, we connect the centers x_i, x_j by a minimizing geodesic E_{ij} , and we define it to be an edge connecting x_i and x_j in Γ . Hence, the total number of edges in Γ is bounded by \tilde{N}^2 .

Note that defined in this way, we may view Γ as a subset of M using the inclusion map. And we may restrict the natural map $f : \Gamma \rightarrow \mathfrak{N}$ to the geodesic graph Γ .

Definition 2.9. A 1-chain α in Γ is a simplicial map $\alpha : [0, 1]_{\Delta} \rightarrow \Gamma$, where $[0, 1]_{\Delta}$ is a simplicial complex obtained by taking a partition $0 < t_1 < \dots < t_L < 1$ of the interval $[0, 1]$ and $L \geq 0$ is an integer. We define the simplicial length $m(\alpha)$ of α to be the number of geodesic segments in α counting with multiplicity. In other words, $m(\alpha) = L + 1$.

Here, we describe a method to estimate the 2-mass of the filling of a 1-cycle in Γ by using the nerve \mathfrak{N} of the covering \mathcal{O} . Suppose C is a 1-cycle in Γ with simplicial length L . We also assume that C bounds a 2-chain in M . Then $f(C)$ is a singular 1-cycle and bounds a singular 2-chain in \mathfrak{N} .

Suppose that x_i is a vertex that is contained in the image of C . Then $f(x_i) \in \mathfrak{N}^{(0)}$, but $f(C)$ may not be in $\mathfrak{N}^{(1)}$. In fact, if $E_{ij} \subset C$ is a minimizing geodesic connecting x_i and x_j , then $B_{R_i}(x_i) \cap B_{R_j}(x_j) \neq \emptyset$. The triangle inequality implies that $E_{ij} \subset B_{R_i}(x_i) \cup B_{R_j}(x_j)$. Hence, the image of $f(E_{ij})$ is a curve in \mathfrak{N} connecting $f(x_i)$ and $f(x_j)$, and $f(E_{ij})$ is contained in a simplex in \mathfrak{N} with vertices $f(x_i)$ and $f(x_j)$. Therefore $f(C)$ is homologous to a simplicial 1-cycle C' in \mathfrak{N} . This implies C' bounds a singular 2-chain in \mathfrak{N} and hence it bounds a simplicial 2-chain in \mathfrak{N} . Moreover, C' has simplicial length L and has the same vertices as $f(C)$.

An important observation in [NR03] is that if a simplicial k -cycle c_k in an n -dimensional simplicial complex with n_0 vertices bounds some simplicial $(k + 1)$ -chain, then it bounds a simplicial $(k + 1)$ -chain of which the size can be estimated using c_k , n_0 and n . More precisely,

Lemma 2.10 ([NR03], Corollary 4.2). *Let $C_k = \sum_i a_k^i \sigma_i^k$ be a simplicial k -cycle in an n -dimensional simplicial complex with n_0 vertices, where $a_k^i \in \mathbb{Z}$ and each σ_i^k being a k -simplex. Suppose that C_k bounds a $(k + 1)$ -chain. Then C_k bounds a simplicial $(k + 1)$ -chain $C_{k+1} = \sum_i a_{k+1}^i \sigma_i^{k+1}$ with $\sum_i |a_{k+1}^i| \leq n_0^{4n \cdot n_0} \max_i |a_k^i|$.*

Therefore, by the above discussion of the nerve \mathfrak{N} , we conclude that

Corollary 2.11. *Let $C_1 = \sum_i a_1^i \sigma_i^1$ be a simplicial 1-cycle in \mathfrak{N} , where $a_1^i \in \mathbb{Z}$ and $\sigma_i^1 : S^1 \rightarrow M$ a Lipschitz map. Then C_1 bounds a 2-chain $C_2 = \sum_i a_2^i \sigma_i^2$ in \mathfrak{N} with $\sum_i |a_2^i| \leq \tilde{N}^{4\tilde{N}+1} \cdot \max_i |a_1^i|$.*

Above Corollary 2.11 implies that C' we constructed before bounds a simplicial 2-chain C'_2 in \mathfrak{N} with no more than $L \cdot \tilde{N}^{4\tilde{N}+1}$ 2-simplices.

Our next step is to find a corresponding 2-chain in M with boundary C . In fact, let us begin a map from the 1-skeleton $\mathfrak{N}^{(1)}$ to M in the following way. If $f(x_i)$ is a 0-simplex in C'_2 , then we map it to x_i . A 1-simplex in C'_2 connecting $f(x_i)$ and $f(x_j)$ is mapped to the minimizing geodesic in Γ connecting x_i and x_j . Therefore, the boundary of a 2-simplex in C'_2 is mapped to geodesic triangles in Γ . The number of geodesic triangles in the image is still bounded by $L \cdot \tilde{N}^{4\tilde{N}+1}$.

Now, if we can fill each geodesic triangle by a 2-chain with controlled 2-mass, then we may obtain a filling of C with bounded 2-mass. Note that a geodesic triangle in Γ is contained in a harmonic ball. Indeed, by the triangle inequality, if x_i, x_j and x_k are vertices of a geodesic triangle, assume that the harmonic radius at x_i is largest among the three points, then the geodesic triangle is contained in $B_{R_i}(x_i) \cup B_{R_j}(x_j) \cup B_{R_k}(x_k) \subset B_{r_h/2}(x_i)$. We show that if the image of a 1-cycle is contained in a harmonic ball, then it bounds a “small” 2-chain inside the ball.

Lemma 2.12. *Suppose $x \in M$ is a point in M . Let $r_h(x)$ be the harmonic radius at x . Then any singular Lipschitz 1-cycle $C \in \mathcal{Z}_1(M, \mathbb{Z})$ whose image of C is contained in the metric ball $B_{r_h(x)/2}(x)$ with $\text{mass}_1(C)$ bounds a singular 2-chain with 2-mass bounded by $\text{mass}_1(C) \cdot r_h(x)$.*

Proof. Let $\Phi : B_{r_h(x)}(0) \rightarrow M$ be the harmonic coordinate at the point $x \in M$, where $\Phi(0) = x$. Suppose that $C = \sum_i a_i f_i$, where $a_i \in \mathbb{Z}$ and $f_i : S^1 \rightarrow M$ being Lipschitz maps. Let $l_i = \text{mass}_1(f_i)$. We show that f_i bounds a 2-chain in $B_{r_h(x)/2}(x)$ with mass bounded by $l_i \cdot r_h(x)$.

We use the same notation f_i to denote the image of the 1-cycle f_i in the metric ball $B_{r_h(x)/2}(x)$. By Lemma 2.2, $\Phi^{-1}(f_i)$ is a piecewise differentiable curve in $B_{r_h(x)}(0) \subset \mathbb{R}^4$ with length bounded by $\sqrt{1+2 \cdot 10^{-3}} \cdot l$. Let us fill $\Phi^{-1}(f_i)$ by a singular 2-chain B_i in $B_{r_h(x)}(0)$, which is obtained by connecting the points on $\Phi^{-1}(f_i)$ with the origin in \mathbb{R}^4 . This 2-chain is a generalized cone with boundary $\Phi^{-1}(f_i)$. The area of this cone is bounded by $\frac{1}{2} \cdot r_h(x) \cdot \text{length}(\Phi^{-1}(f_i)) \leq \frac{\sqrt{1+2 \cdot 10^{-3}}}{2} \cdot l \cdot r_h(x)$. Then $\Phi(B_i)$ defines a 2-chain in $B_{r_h(x)}(x)$ with boundary f_i . By Lemma 2.2, $\text{mass}_2(\Phi(B_i)) \leq \frac{\sqrt{1+2 \cdot 10^{-3}}}{2 \cdot (1-10^{-3})} \cdot l \cdot r_h(x) \leq l \cdot r_h(x)$. \square

3. PROOF OF THEOREM 1.4

Throughout this section, we consider the manifold $M \in \mathcal{M}_1(4, D, v)$. Recall that the idea of the proof of this theorem is the following. We begin with a singular Lipschitz 1-cycle $C \in \mathcal{Z}_1(M; \mathbb{Z})$. Our first step is to find a “nice” representative of the cycle whose image is contained in some body regions. In other words, we are going to show that the cycle C can be written as a sum $\sum C_k$, where the image of each C_k is contained in some body \mathcal{B}_i^j . Then for each C_k , we are going to show that it either bounds a 2-chain in \mathcal{B}_i^j with controlled mass, or it is homologous to a 1-cycle C'_k of which the image is contained in the neck \mathcal{N}_i^{j+1} . We will show that in the later case the cycle $C_k - C'_k$ bounds a 2-chain in \mathcal{B}_i^j and the cycle

We then take a refinement of the above partition. For any integer s , we pick t'_s with $t_s < t'_s < t_{s+1}$ such that $f_k(t'_s) \in \mathcal{B}_i^j - \mathcal{T}_{i_s}^{j+1}$ for s even and $f_k(t'_s) \in \mathcal{T}_{i_s}^{j+1} - \mathcal{B}_i^j$ for s odd. (See Figure 1.)

When s is odd, suppose that $f_k([t_s, t_{s+1}]) \subset \mathcal{T}_{i_s}^{j+1}$. Because $\mathcal{N}_{i_s}^{j+1}$ is connected, both $f_k(t_s)$ and $f_k(t_{s+1})$ are contained in $\mathcal{N}_{i_s}^{j+1}$. In this case, we connect $f_k(t_s)$ and $f_k(t_{s+1})$ by a minimizing geodesic γ_s in $\mathcal{N}_{i_s}^{j+1}$. (See Figure 2.)

By Lemma 2.8, (also see Figure 3), we have

$$\text{length}(\gamma_s) \leq \text{diam}(\mathcal{N}_{i_s}^{j+1}) \leq B(v) \cdot \text{thick}(\mathcal{N}_{i_s}^{j+1}) \leq B(v) \cdot \text{length}(f_k([t'_s, t'_{s+1}])).$$

Therefore, we have an estimation of the sum of the length of curves γ_s given by

$$\sum_{s \text{ odd}} \text{length}(\gamma_s) \leq B(v) \cdot \sum_{s \text{ odd}} \text{length}(f_k([t'_s, t'_{s+1}])) \leq B(v) \cdot \text{length}(f_k).$$

When s is even, by construction, the arc $f_k([t_s, t_{s+1}]) \subset \mathcal{B}_i^j$. Therefore, the image of the curve $f_k([t_0, t_1]) \cup \gamma_1 \cup \dots \cup \gamma_{n-1} \cup f_k([t_n, t_{n+1}])$ is contained in \mathcal{B}_i^j . We take C_j^m to be a 1-cycle whose image is this curve. (See Figure 4.)

When s is odd, the arc $f_k([t_s, t_{s+1}]) \subset \mathcal{T}_{i_s}^{j+1}$ and $f_k([t_s, t_{s+1}]) \cup (\gamma_s)$ is a closed curve. For a fixed i , we take $C_{j+1,i}^m$ to be a cycle whose image is the union of all $f_k([t_l, t_{l+1}]) \cup (\gamma_l)$'s such that $f_k([t_l, t_{l+1}]) \subset \mathcal{T}_i^{j+1}$.

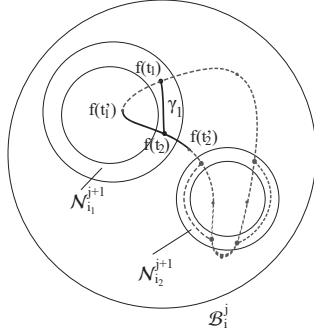


FIGURE 3. $\text{length}(\gamma_1) \leq B(v) \cdot \text{length}(f([t'_1, t'_2]))$.

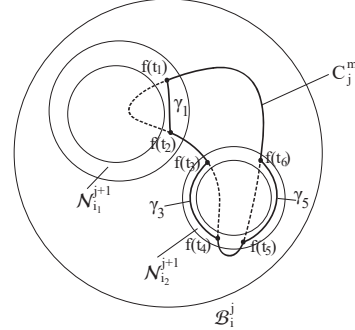


FIGURE 4. Choosing C_j^m .

Then f_k can be represented by $C_j^k + \sum_{i \in I} C_{j+1,i}^k$, for some index set I and

$$\text{mass}_1(C_j^k) + \sum_{i \in I} \text{mass}_1(C_{j+1,i}^k) \leq \text{length}(f_k) + 2 \sum_{s \text{ odd}} \text{length}(\gamma_s) \leq (2B(v) + 1) \cdot \text{length}(f_k).$$

We then apply the same construction to each $C_{j+1,i}^k$ and to every k . Now for every fixed \mathcal{B}_i^j , let $C_{i,j}$ be the cycle whose image is the union of all $C_{m,l}^k$'s such that the image of $C_{m,l}^k \subset \mathcal{B}_i^j$.

Then $C = \sum_{i,j \in J} C_{i,j}$, for some index set J . The size of J is bounded by the total number of the bodies $n(v, D) \leq N(v, D)k(v, D)$. And the mass

$$\sum_{i,j \in J} \text{mass}_1(C_{i,j}) \leq (2B(v) + 1)^{k(v,D)} \text{mass}_1(C),$$

□

As described above, our next step is to show that for any Lipschitz 1-cycle C whose image is contained in a single body \mathcal{B}_i^j , it either bounds a “small” 2-chain whose image is contained in \mathcal{T}_i^j , or it is homologous to some cycle in \mathcal{N}_i^j . And the idea is to apply certain finiteness property of the geodesic graph Γ we constructed in Section 2.2. To begin with, we first show that the cycle C is homologous to a cycle C' in subgraph of Γ . The centers of this subgraph are in B_i^j . The mass of this cycle C' is comparable with the mass of the cycle C .

Lemma 3.2. *For any singular Lipschitz 1-cycle C with its image contained in the body \mathcal{B}_i^j , C is homologous to a 1-cycle C' in Γ . The image of C' is also contained in \mathcal{B}_i^j . And the simplicial length of C' is bounded by $\tilde{N}^2 \cdot \text{mass}_1(C)/R_i^j$, where $R_i^j = r_{h_i^j}/20$ as we defined in Section 2.2. Furthermore,*

$$\text{mass}_1(C') \leq 2\tilde{N}^2 \cdot \text{mass}_1(C).$$

There is a singular 2-chain E with $\partial E = C - C'$ and

$$\text{mass}_2(E) \leq 20R_i^j \cdot (2\tilde{N}^2 + 3) \cdot \text{mass}_1(C').$$

Proof. Suppose that $C = \sum a_k f_k$, where $a_k \in \mathbb{Z}$ and $f_k : S^1 \rightarrow M$ is Lipschitz. Let us consider each map f_k . We parameterize the map f_k by $f_k : [0, 1] \rightarrow M$. Let us take a partition $0 = t_0 < t_1 < \dots < t_L = 1$ of the interval $[0, 1]$ such that $\text{length}(f_k(t_m, t_{m+1})) = R_i^j$. We have

$$L \leq \text{mass}_1(f_k)/R_i^j.$$

Suppose that $f_k(t_m) \in B_{R_i^j}(x)$ and $f_k(t_{m+1}) \in B_{R_i^j}(y)$ for some $x, y \in \mathcal{B}_i^j$. Denote by σ_m a minimizing geodesic connecting $f_k(t_m)$ and x , and σ_{m+1} a minimizing geodesic connecting $f_k(t_{m+1})$ and y . Note that by the triangle inequality, the distance between x and y is less than $3R_i^j$, so $B_{R_i^j}(x) \cap B_{R_i^j}(y) = \emptyset$.

Our next step is to construct a 1-chain α_m in Γ with boundary x and y so that the 1-cycle $\gamma_m := -\alpha_m \cup \sigma_m \cup f_k(t_m, t_{m+1}) \cup (-\sigma_{m+1})$ is contained in $B_{r_{h_i^j}(x)/2}(x)$. The 1-chain α_m is constructed in following way. We partite the interval $[t_i, t_{i+1}]$ further into $t_i = t_{i,0} < t_{i,1} < \dots < t_{i,u} = t_{i+1}$ such that if $f_k(t_{i,v}) \in B_{R_i^j}(x_v)$ and $f_k(t_{i,v+1}) \in B_{R_i^j}(x_{v+1})$, then $B_{R_i^j}(x_v) \cap B_{R_i^j}(x_{v+1}) \neq \emptyset$. Note that those x_v may repeat. There is a 1-chain in Γ connecting x and y through centers x_v 's. In other words there is a continuous map $\beta_m : [0, 1] \rightarrow \Gamma$ and $0 = s_0 < \dots < s_u = 1$, such that $\beta_m(0) = x$ and $\beta_m(1) = y$ and $\beta_m(s_v) = x_v$.

If $\beta_m(s_v) = \beta_m(s_{v'})$, we delete $\beta_m([s_v, s_{v'}])$. We define α_m to be the resulting a 1-chain after we deleted all the repetitions in β_m . Therefore, the simplicial length of α_m is bounded by \tilde{N}^2 , and hence the 1-mass of α_m is bounded by $2R_i^j \cdot \tilde{N}^2$. And therefore,

$$\text{mass}_1(\gamma_m) \leq 2R_i^j \cdot \tilde{N}^2 + 3R_i^j.$$

Now by Lemma 2.12, each γ_m bounds a 2-chain E_m with 2-mass bounded by

$$\text{mass}_2(E_m) \leq r_{h_i^j} \cdot R_i^j \cdot (2\tilde{N}^2 + 3) \leq 20R_i^j \cdot (2\tilde{N}^2 + 3) \cdot R_i^j.$$

Let $\alpha = \sum_m \alpha_m$. The simplicial length of α is bounded by $\tilde{N}^2 \cdot L \leq \tilde{N}^2 \cdot \text{mass}_1(f_k)/R_i^j$ and hence the 1-mass of α is bounded by $2R_i^j \tilde{N}^2 \cdot L \leq 2\tilde{N}^2 \cdot \text{mass}_1(f_k)$.

Note that $\sum_m \gamma_m = f_k - \alpha$. Therefore, we conclude that α is homologous to f_k and the cycle $\sum_m \gamma_m = f_k - \alpha$ bounds a 2-chain $\sum_m E_m$ with 2-mass bounded by $20R_i^j \cdot (2\tilde{N}^2 + 3) \cdot$

$\text{mass}_1(\alpha)$. Now we apply this construction to all components f_k 's and let C' be the sum of all corresponding α 's, we conclude that C' satisfies the desired properties. \square

Now we are going to show that a cycle C whose image is contained in \mathcal{B}_i^j either bounds a 2-chain in \mathcal{T}_i^j , or it can be “move out” from \mathcal{B}_i^j . In the later case the cycle C is homologous to a cycle whose image is contained in the neck region $\mathcal{N}_i^j \subset \mathcal{B}_i^j \cap \mathcal{B}_{i'}^{j-1}$ for some body $\mathcal{B}_{i'}^{j-1}$.

Lemma 3.3. *Let C be a singular Lipschitz 1-cycle with its image contained in a body \mathcal{B}_i^j . Then either C bounds a singular 2-chain in \mathcal{T}_i^j or C is homologous to a singular Lipschitz 1-cycle C'' such that*

- (1) *The image of C'' is contained in $\mathcal{N}_i^j \subset \mathcal{B}_i^j \cap \mathcal{B}_{i'}^{j-1}$.*
- (2) *There is a singular 2-chain E with its image contained in \mathcal{T}_i^j such that $\partial E = C - C''$.*
- (3) *$\text{mass}_1(C'') \leq h(v) \cdot \text{diam}(\mathcal{N}_i^j)$, for some function h which only depends on v and $\text{diam}(\mathcal{N}_i^j)$ is the diameter of a neck \mathcal{N}_i^j as a manifold with pullback metric.*

Proof. By our assumption, the first homology $H^1(M) = 0$. Hence the 1-cycle C bounds a 2-chain in M . Suppose the image of this 2-chain is not contained in \mathcal{T}_i^j . Then C is homologous to a Lipschitz 1-cycle C' in \mathcal{N}_i^j . To see this, one can take a small perturbation of the image of the 2-chain so that the intersection between the 2-chain and boundary of \mathcal{N}_i^j is transverse and we define C' to a cycle whose image is given by this intersection.

Note that since \mathcal{N}_i^j is diffeomorphic to $\mathbb{R} \times S^3/\Gamma_i^j$ for some $\Gamma_i^j \subset O(3)$, the fundamental group $\pi_1(\mathcal{N}_i^j) \cong \Gamma_i^j$ is finite. In particular, the size of the first homology group $|H_1(\mathcal{N}_i^j)| \leq |\Gamma_i^j| \leq h_1(v)$, where h_1 is a function which only depends on v . By Gromov Lemma [Gro07], the generators of $H_1(\mathcal{N}_i^j)$ can be represented by curves of length bounded by $2 \text{diam}(\mathcal{N}_i^j)$. Since $H_1(\mathcal{N}_i^j)$ is finite abelian group with at most $h_1(v)$ generators, one can choose a representative C'' in the class $[C']$ such that $\text{mass}_1(C'') \leq 2h_1(v)^{h_1(v)} \text{diam}(\mathcal{N}_i^j) := h(v) \text{diam}(\mathcal{N}_i^j)$. \square

Remark 3.4. In the above Lemma, we have no control of the size of the 2-chain E . In fact the 2-mass of the 2-chain E can be large since we are trying to find a “nice” representative of the class $[C']$.

We are going to show below in Lemma 3.5 and Lemma 3.6 that in both two cases of Lemma 3.3, one can always construct a “new” 2-chain with boundary C such that the 2-mass of the chain is bounded by a function that only depends on v , D and the 1-mass of C .

Lemma 3.5. *Let C be a singular Lipschitz 1-cycle whose image is contained in \mathcal{B}_i^j . If C bounds a singular 2-chain whose image is contained in \mathcal{T}_i^j , then C bounds a singular 2-chain E in M such that $\text{mass}_2(E) \leq g_1(v, D) \cdot \text{mass}_1(C)$, for some function g_1 that only depends on v and D .*

Proof. Suppose that C bounds a singular 2-chain in \mathcal{T}_i^j . By Lemma 3.2, C is homologous to a 1-cycle C' in Γ . The image of C' is contained in \mathcal{B}_i^j and the simplicial length of C' is bounded by $2\tilde{N}^2 \cdot \text{mass}_1(C)/R_i^j$. Furthermore, there is a singular 2-chain G such that $\partial G = C - C'$ and

$$\text{mass}_2(G) \leq 20R_i^j \cdot (2\tilde{N}^2 + 3) \cdot \text{mass}_1(C) \leq 20D \cdot (2\tilde{N}^2 + 3) \cdot \text{mass}_1(C).$$

Because C and C' are homologous, the cycle C' also bounds a 2-chain in \mathcal{T}_i^j . If we apply the method in Section 2.2 and Corollary 2.11 to $\mathfrak{N}(\mathcal{T}_i^j)$, the 1-cycle C' bounds a 2-chain F

in \mathcal{T}_j^i . The image of the boundary of F consists of at most $2\tilde{N}^{4\tilde{N}^{\tilde{N}+1}}\tilde{N}^2 \text{mass}_1(C)/R_i^j$ geodesic triangles in Γ .

Note that these geodesic triangles are also in \mathcal{T}_j^i . And if $\mathcal{B}_{i'}^{j'}$ is a child of \mathcal{B}_i^j , then $R_{i'}^{j'} < R_i^j$. Therefore the circumference of each geodesic triangle is at most $3 \cdot 2R_i^j$. Hence each geodesic triangle can be contained in a harmonic ball of radius $20R_i^j$. By Lemma 2.12, one can fill each geodesic triangle by a 2-chain with area at most $6 \cdot 20(R_i^j)^2$. In other words, the 2-mass of F is bounded by $240\tilde{N}^{4\tilde{N}^{\tilde{N}+1}} \cdot \tilde{N}^2 \cdot \text{mass}_1(C) \cdot R_i^j \leq 240D\tilde{N}^{4\tilde{N}^{2\tilde{N}}} \cdot \text{mass}_1(C)$. Let $E = F' + G$. Then $\partial E = \partial(F' + G) = C - C' + C' = C$ and

$$\text{mass}_2(E) \leq (240\tilde{N}^{4\tilde{N}^{2\tilde{N}}} + 40\tilde{N}^2 + 60)D \text{mass}_1(C).$$

□

Lemma 3.6. *Let C be a singular Lipschitz 1-cycle in \mathcal{B}_i^j . If C is homologous a singular 1-cycle C' in \mathcal{N}_i^j as described in Lemma 3.3, then C bounds a singular 2-chain E in M such that $\text{mass}_2(E) \leq g_1(v, D) \cdot \text{mass}_1(C) + g_2(v, D)f(v)$, for some function g_1, g_2 and f .*

Proof. By Lemma 3.3, the 1-mass of the 1-cycle satisfies $\text{mass}_1(C') \leq h(v) \text{diam}(\mathcal{N}_i^j)$. Since \mathcal{N}_i^j is covered by no more than \tilde{N} balls of radius R_i^j , $\text{diam}(\mathcal{N}_i^j) \leq \tilde{N} \cdot R_i^j \leq \tilde{N}D$.

Therefore, $\text{mass}_1(C') \leq h(v) \cdot \tilde{N}R_i^j \leq h(v) \cdot \tilde{N}D$. And by Lemma 3.2, C is homologous to C'' in \mathcal{B}_j^i . C'' is a 1-cycle in Γ such that $\text{mass}_1(C'') \leq 2R_i^j\tilde{N}^3h(v) \leq 2D\tilde{N}^3h(v)$ and its simplicial length is bounded by $\tilde{N}^3h(v)$. Therefore, by Lemma 3.5, the cycle $C - C''$ bounds a 2-chain G such that $\text{mass}_2(G) \leq g_1(v, D)(\text{mass}_1(C) + \text{mass}_1(C''))$.

Now because $H^1(M)$ is trivial, the cycle C'' bounds a 2-chain in M . We apply the method in Section 2.2 and Corollary 2.11 to C'' to conclude that the cycle C'' bounds a 2-chain F in M such that the image of the boundary of F consists of at most $\tilde{N}^{4\tilde{N}^{\tilde{N}+1}}\tilde{N}^3h(v)$ geodesic triangles in Γ . The length of each edge is at most D and the triangle is contained in a harmonic ball of radius at most D . By Lemma 2.12 Each triangle can be filled by a 2-chain with 2-mass bounded by $3D^2$. Therefore, we obtain a 2-chain F with $\partial F = C''$ and

$$\text{mass}_2(F) \leq 3\tilde{N}^{4\tilde{N}^{\tilde{N}+1}}\tilde{N}^3 \cdot D^2 \cdot h(v).$$

Let $E = G + F$. Then $\partial E = C - C'' + C'' = C$ and

$$\text{mass}_2(E) \leq g_1(v, D) \text{mass}_1(C) + g_2(v, D)h(v),$$

where

$$g_1(v, D) = (240\tilde{N}^{4\tilde{N}^{2\tilde{N}}} + 40\tilde{N}^2 + 60)D,$$

and

$$g_2(v, D) = 3\tilde{N}^{4\tilde{N}^{\tilde{N}+1}}\tilde{N}^3 \cdot D^2 + g_1(v, D) \cdot 2D\tilde{N}^3.$$

□

We may now complete the proof of Theorem 1.4.

Proof of Theorem 1.4. Let $C \in \mathcal{Z}_1(M, \mathbb{Z})$ be a singular Lipschitz 1-cycle. By Lemma 3.1, we can find a representation of $C = \sum_{k=1}^n (v, D)C_k$, where $n(v, D) \leq N(v, D) \cdot k(v, D)$ and each C_k is a Lipschitz 1-cycle with image contained in some body \mathcal{B}_j^i and the total mass satisfies

$$\sum_{k=1}^n \text{mass}_1(C_k) \leq (2B(v) + 1)^{k(v, D)} \cdot \text{mass}_1(C).$$

Now by Lemma 3.3, each C_k either bounds a 2-chain in \mathcal{T}_j^i , or it is homologous to some C'_k such that the image of C'_k is contained in $\mathcal{N}_j^i \subset \mathcal{B}_j^i \cap \mathcal{B}_j^{i-1}$.

In the first case, Lemma 3.5 tells us that C_k bounds a 2-chain E_k with

$$\text{mass}_2(E_k) \leq g_1(v, D) \text{mass}_1(C_k).$$

And in the second case, we apply Lemma 3.6 to obtain a 2-chain E_k with

$$\text{mass}_2(E_k) \leq g_1(v, D) \text{mass}_1(C_k) + g_2(v, D)h(v).$$

If we take the sum over all n , $\sum_{k=1}^n C_k$ bounds 2-chain with 2-mass bounded by

$$g_1(v, D) \cdot \sum_{k=1}^n \text{mass}_1(C_k) + g_2(v, D) \cdot n(v, D) \cdot h(v).$$

Therefore we conclude that C bounds 2-chain with 2-mass bounded by

$$f_1(v, D) \text{mass}_1(C) + f_2(v, D),$$

where

$$f_1(v, D) = g_1(v, D) \cdot (2B(v) + 1)^{k(v, D)}$$

and

$$f_2(v, D) = g_2(v, D) \cdot h(v) \cdot N(v, D) \cdot k(v, D).$$

□

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REFERENCES

- [AC91] Michael T Anderson and Jeff Cheeger. Diffeomorphism finiteness for manifolds with ricci curvature and l^2 -norm of curvature bounded. *Geometric & Functional Analysis GAFA*, 1(3):231–252, 1991.
- [And89] Michael T Anderson. Ricci curvature bounds and einstein metrics on compact manifolds. *Journal of the American Mathematical Society*, 2(3):455–490, 1989.
- [And92] Michael T Anderson. The l^2 structure of moduli spaces of einstein metrics on 4-manifolds. *Geometric and Functional Analysis*, 2(1):29–89, 1992.
- [CC96] Jeff Cheeger and Tobias H Colding. Lower bounds on ricci curvature and the almost rigidity of warped products. *Annals of mathematics*, 144(1):189–237, 1996.
- [CC97] Jeff Cheeger and Tobias H Colding. On the structure of spaces with ricci curvature bounded below. i. *Journal of Differential Geometry*, 46(3):406–480, 1997.
- [CN14] Jeff Cheeger and Aaron Naber. Regularity of einstein manifolds and the codimension 4 conjecture. *arXiv preprint arXiv:1406.6534*, 2014.
- [Col97] Tobias H Colding. Ricci curvature and volume convergence. *Annals of mathematics*, 145(3):477–501, 1997.
- [Coo15] Michel Coornaert. *Topological dimension and dynamical systems*. Springer, 2015.
- [Fed14] Herbert Federer. *Geometric measure theory*. Springer, 2014.
- [Gro83] Mikhail Gromov. Filling riemannian manifolds. *Journal of Differential Geometry*, 18(1):1–147, 1983.

- [Gro07] Mikhail Gromov. *Metric structures for Riemannian and non-Riemannian spaces*. Springer Science & Business Media, 2007.
- [Lic62] WB Raymond Lickorish. A representation of orientable combinatorial 3-manifolds. *Annals of Mathematics*, pages 531–540, 1962.
- [NR03] Alexander Nabutovsky and Regina Rotman. Upper bounds on the length of a shortest closed geodesic and quantitative hurewicz theorem. *Journal of the European Mathematical Society*, 5(3):203–244, 2003.
- [NR06] Alexander Nabutovsky and Regina Rotman. Curvature-free upper bounds for the smallest area of a minimal surface. *Geometric & Functional Analysis GAFA*, 16(2):453–475, 2006.